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The characterization of two-component (2+1)-dimensional integrable systems of hydrodynamic type

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Abstract
We obtain the necessary and sufficient conditions for a two-component (2+1)-dimensional system of hydrodynamic type to possess infinitely many hydrodynamic reductions. These conditions are in involution, implying that the systems in question are locally parametrized by 15 arbitrary constants. It is proved that all such systems possess three conservation laws of hydrodynamic type and, therefore, are symmetrizable in Godunov's sense. Moreover, all such systems are proved to possess a scalar pseudopotential which plays the role of the 'dispersionless Lax pair'. We demonstrate that the class of two-component systems possessing a scalar pseudopotential is in fact *identical* with the class of systems possessing infinitely many hydrodynamic reductions, thus establishing the equivalence of the two possible definitions of integrability. Explicit linearly degenerate examples are constructed.

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1. Introduction

The last years were marked by a remarkable progress in the theory of one-dimensional systems of hydrodynamic type,

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x = 0$$

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which naturally occur in applications in gas dynamics, hydrodynamics, chemical kinetics, Whitham averaging procedure, differential geometry and topological field theory. We refer to the reviews [6, 7, 33, 35] for further discussion and references. It has been observed that many particularly important examples are diagonalizable, i.e. reducible to the Riemann invariant form

$$R^i_t + v^i(R)R^i_x = 0$$

where the characteristic speeds $v^i(R)$ satisfy the so-called semi-Hamiltonian property [35] (also known as the 'richness' condition [33]),

$$\partial_k\left(\frac{\partial_j v^i}{v^j - v^i}\right) = \partial_j\left(\frac{\partial_k v^i}{v^k - v^i}\right)$$

for any triple $i \neq j \neq k$. Semi-Hamiltonian systems possess infinitely many conservation laws and commuting flows of hydrodynamic type, and can be linearized by the generalized hodograph transform [35]. Their analytic, differential-geometric and Hamiltonian aspects are well-understood by now.

In contrast, not much was known about the integrability of multi-dimensional systems of hydrodynamic type until recently. The main problem is that the standard symmetry approach to the integrability based on the existence of higher symmetries (conservation laws) does not seem to be effective in this context. In this paper we consider the problem of characterization of (2+1)-dimensional integrable quasilinear systems

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y = 0 \tag{1}$$

where t, x, y are independent variables, $\mathbf{u}$ is an m-component column vector and $A(\mathbf{u})$, $B(\mathbf{u})$ are $m \times m$ matrices. We assume that the system is strictly hyperbolic, i.e. the generic matrix of the linear family $\lambda I_m + \mu A + B$ has m distinct real eigenvalues. Following our recent paper [14], we call the system (1) *integrable* if it possesses 'sufficiently many' exact solutions of the form $\mathbf{u} = \mathbf{u}(R^1, \ldots, R^n)$ where the *Riemann invariants* $R^1, \ldots, R^n$ solve a pair of commuting diagonal systems

$$R^i_t = \lambda^i(R)R^i_y \qquad R^i_x = \mu^i(R)R^i_y \tag{2}$$

we emphasize that the number n of Riemann invariants is allowed to be arbitrary. Solutions of this type, known as nonlinear interactions of n planar simple waves, were discussed in multi-dimensional hydrodynamics and magnetohydrodynamics in a series of publications [3, 4, 20, 32]. Later, they were investigated by Gibbons and Tsarev in the context of the dispersionless KP hierarchy [15–18], (see also [28]), and the theory of Egorov's integrable hydrodynamic chains [30, 31]. The interpretation of n-wave interactions as symmetry constraints was proposed in [2].

Particularly important examples of integrable multi-dimensional systems of hydrodynamic type and dispersionless PDEs related to them arise in general relativity, differential geometry (the theory of Einstein–Weyl spaces [8, 9]), in the context of the Dirichlet boundary problem in multi-connected domains [27] and the Whitham averaging procedure (in particular, the dispersionless limit) applied to (2+1)-dimensional solitonic PDEs [25, 26, 36]. All known integrable examples turn out to be conservative (see section 3 where we prove that this is always the case, at least in the two-component situation) and, moreover, possess exactly one 'extra' conservation law which is the necessary ingredient of the theory of weak solutions. The property of integrability allows one to construct infinitely many (implicit) solutions and investigate their breakdown and singularity structure. This makes the class of integrable multi-dimensional quasilinear systems a possible venue for developing and testing the mathematical theory (existence, uniqueness, weak solutions, etc) of multidimensional conservation laws which, currently, remain *terra incognita* [5].

We recall (see [35]) that the requirement of the commutativity of the flows (2) is equivalent to the following restrictions on their characteristic speeds:

$$\frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i} \qquad i \neq j \quad \partial_j = \partial/\partial_{R^j} \tag{3}$$

(no summation!). Once these conditions are met, the general solution of (2) is given by the implicit 'generalized hodograph' formula [35]

$$v^i(R) = y + \lambda^i(R)t + \mu^i(R)x \qquad i = 1, \ldots, n \tag{4}$$

where $v^i(R)$ are characteristic speeds of the general flow commuting with (2), i.e. the general solution of the linear system

$$\frac{\partial_j v^i}{v^j - v^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} = \frac{\partial_j \mu^i}{\mu^j - \mu^i}. \tag{5}$$

Substituting $\mathbf{u}(R^1, \ldots, R^n)$ into (1) and using (2), one readily arrives at the equations

$$(\lambda^i I_m + \mu^i A + B)\partial_i \mathbf{u} = 0 \qquad i = 1, \ldots, n \tag{6}$$

implying that λ^i and μ^i satisfy the dispersion relation

$$\det(\lambda I_m + \mu A + B) = 0. \tag{7}$$

Thus, the construction of nonlinear interactions of n planar simple waves reduces to solving equations (3), (6) for $\mathbf{u}(R)$, $\lambda^i(R)$, $\mu^i(R)$ as functions of the Riemann invariants $R^1, \ldots, R^n$. For $n \geqslant 3$ these equations are highly overdetermined and do not possess solutions in general. As demonstrated in [14], the requirement of the existence of nontrivial three-component reductions is very restrictive and implies, in particular, the existence of n-component reductions for arbitrary n. We give the following

Definition. *System (1) is said to be integrable if it possesses n-component reductions of the form (2) parametrized by n arbitrary functions of a single variable.*

We refer to [14] for the motivation and supporting examples.

Remark 1. In the case of linear systems (1), i.e. when both A and B are constant matrices, equations (3) and (7) imply $\lambda^i_j = \mu^i_j = 0$, so that $\lambda^i = \lambda^i(R^i)$, $\mu^i = \mu^i(R^i)$. Moreover, as follows from (6), $\partial_i u = \xi_i(R^i)$ where $\xi_i(R^i)$ is the right eigenvector of the matrix $\lambda^i I_m + \mu^i A + B$. With the particular choice $\lambda^i = \text{const}, \mu^i = \text{const}, \xi_i = \text{const}$ the corresponding solutions represent the standard linear superposition of simple waves, $u = \sum f^i(x + \lambda^i t + \mu^i y)\xi_i$.

In section 2 we derive the integrability conditions for the two-component system (1) assuming that the matrix A is written in the diagonal form,

$$\begin{pmatrix} v \\ w \end{pmatrix}_t + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_x + \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}_y = 0 \tag{8}$$

such diagonalization is always possible in the two-component situation. These conditions constitute a complicated overdetermined system (13)–(17) of second-order PDEs for a, b, p, q, r, s as functions of v, w, which is in involution; a simple analysis shows that the class of integrable two-component systems is locally parametrized by 15 arbitrary constants.

Remark 2. In principle, the method described in section 2 allows one to derive the integrability conditions in arbitrary coordinates; however, the formulae become extremely complicated. We were not able to find an invariant 'tensor' formulation of the integrability conditions so far.

We prove (theorem 1 of section 3) that an arbitrary two-component system (8) satisfying the integrability conditions possesses three conservation laws of hydrodynamic type and, thus, is symmetrizable in Godunov's sense [19].

In section 4 we demonstrate that all two-component integrable systems possess scalar pseudopotentials of the form

$$\psi_t = f(\psi_y, v, w) \qquad \psi_x = g(\psi_y, v, w).$$

According to the philosophy of [36], this indicates that (2+1)-dimensional integrable systems of hydrodynamic type can be obtained as dispersionless limits from the appropriate (2+1)-dimensional integrable soliton equations (possibly, nonlocal, differential-difference, etc). The corresponding pseudopotentials are quasiclassical limits of the associated linear Lax operators. The construction of the 'solitonic prototype' was sketched in the case when the dependence of f and g on ψ_y is rational (trigonometric), leading to differential (difference) soliton equations. We prove (theorem 2 of section 4) that the requirement of the existence of a scalar pseudopotential is, in fact, necessary and sufficient for the existence of the infinity of hydrodynamic reductions. This establishes the equivalence of the two approaches to the integrability of (2+1)-dimensional hydrodynamic-type systems. The quasi-classical $\bar{\partial}$-dressing approach to the solution of (2+1)-dimensional dispersionless systems based on the pseudopotentials of the above type was proposed in the series of recent publications [1, 21–24]. The interpretation of exact solutions describing nonlinear interactions of planar simple waves as symmetry constraints was outlined in the recent publication [2].

Some explicit examples where the matrix A is linearly degenerate (i.e., $a_v = b_w = 0$) are discussed in section 5. These include a remarkable case where *both* matrices A and B, as well as arbitrary linear combinations thereof, are linearly degenerate.

We conclude this introduction by listing some known examples of two-component integrable systems written in the form (8).

Example 1. Let us consider the system

$$v_t + \frac{1}{v+w} v_x - \frac{1}{v+w} w_y = 0 \qquad w_t - \frac{1}{v+w} w_x + \frac{1}{v+w} v_y = 0$$

here $a = \frac{1}{v+w}$, $b = -\frac{1}{v+w}$, etc. Introducing the variables $m = v + w$, $n = v - w$, one can rewrite these equations in the form

$$(\partial_x + \partial_y)n + \tfrac{1}{2}\partial_t m^2 = 0 \qquad \partial_t n + (\partial_x - \partial_y)\ln m = 0$$

leading, upon cross-differentiation, to the Boyer–Finley equation for $m^2 = (v + w)^2$

$$\partial_t^2 m^2 = \left(\partial_x^2 - \partial_y^2\right) \ln m^2.$$

The Boyer–Finley equation is known to be integrable, its hydrodynamic reductions were investigated, e.g., in [12].

Example 2. A closely related example is

$$v_t + \frac{1}{v+w} v_x + \frac{1}{v+w}\sqrt{\frac{v}{w}}\, w_y = 0 \qquad w_t - \frac{1}{v+w} w_x + \frac{1}{v+w}\sqrt{\frac{w}{v}}\, v_y = 0.$$

Note that the characteristic speeds a and b are the same as in the previous example! In the new variables $m = v - w$, $n = 2\sqrt{vw}$, this system reduces to

$$m_t + \frac{mm_x + nn_x}{m^2 + n^2} + \frac{mn_y - nm_y}{m^2 + n^2} = 0 \qquad n_t + \frac{nm_x - mn_x}{m^2 + n^2} + \frac{mm_y + nn_y}{m^2 + n^2} = 0.$$

In this form it has appeared in a recent paper [14]. It was demonstrated, in particular, that the expression $\rho^2 = m^2 + n^2 = (v + w)^2$ satisfies another version of the Boyer–Finley equation,

$$\partial_t^2 \rho^2 = \left(\partial_x^2 + \partial_y^2\right) \ln \rho^2$$

corresponding to a different signature.

Example 3. Here both matrices A and B, as well as arbitrary linear combinations thereof, are linearly degenerate:

$$v_t + w v_x + \frac{1}{w - v}(v_y + w_y) = 0 \qquad w_t + v w_x + \frac{1}{v - w}(v_y + w_y) = 0.$$

Introducing the variables $m = v + w$, $n = vw$, one can rewrite these equations as

$$m_t + n_x = 0 \qquad n_t + m n_x - n m_x + m_y = 0.$$

This system, which is descriptive of hyperCR Einstein–Weyl structures [9], was thoroughly investigated in [30] (see also [29, 34]).

Example 4. We also looked at the integrable systems (8) whose chararacteristic speeds a and b are of the form $a = v + w + \epsilon v$, $b = v + w + \epsilon w$, $\epsilon =$ const. The analysis showed that the only possible values for ϵ are $\epsilon = -1$ and $\epsilon = -2$. In the first case the matrix A is linearly degenerate, see section 4 for the general form of the corresponding matrix B. In the case $\epsilon = -2$ we obtained the system

$$v_t + (v - w)v_x + w_y = 0 \qquad w_t + (w - v)w_x + v_y = 0$$

which is yet another first-order form of the Boyer–Finley equation; indeed, this system reduces to that from example 1 after a simple change of variables $w \to -w$, $t \leftrightarrow x$.

2. Derivation of the integrability conditions

The integrability conditions can be obtained as follows. Looking for reductions of the system (8) in the form $v = v(R^1, \ldots, R^n)$, $w = w(R^1, \ldots, R^n)$ where the Riemann invariants satisfy equations (2), and substituting into (8), one arrives at

$$(\lambda^i + a\mu^i + p)\partial_i v + q\partial_i w = 0 \qquad r\partial_i v + (\lambda^i + b\mu^i + s)\partial_i w = 0$$

(no summation!) so that λ^i and μ^i satisfy the dispersion relation

$$(\lambda^i + a\mu^i + p)(\lambda^i + b\mu^i + s) = qr.$$

We assume that the dispersion relation defines an irreducible conic, i.e. $a \neq b$, $r \neq 0$, $q \neq 0$. Note that these conditions are equivalent to the requirement $rk[A, B] = 2$. Setting $\partial_i v = \varphi^i \partial_i w$ one obtains the following expressions for λ^i and μ^i in terms of φ^i,

$$\lambda^i = \frac{ar(\varphi^i)^2 + (as - bp)\varphi^i - bq}{(b - a)\varphi^i} \qquad \mu^i = \frac{r(\varphi^i)^2 + (s - p)\varphi^i - q}{(a - b)\varphi^i} \tag{9}$$

which define a rational parametrization of the dispersion relation. The compatibility conditions of the equations $\partial_i v = \varphi^i \partial_i w$ imply

$$\partial_i \partial_j w = \frac{\partial_j \varphi^i}{\varphi^j - \varphi^i}\partial_i w + \frac{\partial_i \varphi^j}{\varphi^i - \varphi^j}\partial_j w \tag{10}$$

while the commutativity conditions (3) lead to the expressions for $\partial_j \varphi^i$, $(i \neq j)$, in the form

$$\partial_j \varphi^i = (\cdots)\partial_j w. \tag{11}$$

Here dots denote a rational expression in φ^i, φ^j whose coefficients depend on a, b, p, q, r, s and the first derivatives thereof. We do not write them out explicitly due to their complexity. To manipulate with these expressions we used symbolic computations (Mathematica 5.0). Substituting the expressions for $\partial_j \varphi^i$ into (10) one obtains

$$\partial_i \partial_j w = (\cdots) \partial_i w \partial_j w \tag{12}$$

where, again, dots denote a rational expression in φ^i, φ^j. One can see that the compatibility conditions of equations (11), i.e. $\partial_k \partial_j \varphi^i - \partial_j \partial_k \varphi^i = 0$ are of the form $P \partial_j w \partial_k w = 0$, where P is a complicated rational expression in $\varphi^i, \varphi^j, \varphi^k$ whose coefficients depend on partial derivatives of a, b, p, q, r, s up to second order (to obtain the integrability conditions it suffices to consider three-component reductions setting $i = 1, j = 2, k = 3$). Requiring that P vanishes identically we obtain the expressions for all second partial derivatives of the potentials a, b, p, s, as well as three relations among the second partial derivatives of q and r. Similarly, the compatibility conditions of equations (12), i.e. $\partial_k(\partial_i \partial_j w) - \partial_j(\partial_i \partial_k w) = 0$ take the form $Q \partial_i w \partial_j w \partial_k w = 0$ where, again, Q is a rational expression in $\varphi^i, \varphi^j, \varphi^k$. Equating Q to zero, one obtains (modulo conditions obtained in the previous step) the expressions for mixed partial derivatives q_{vw} and r_{vw}. The resulting set of the integrability conditions looks as follows.

Equations for a

$$\begin{aligned} a_{vv} &= \frac{q a_v b_v + 2q a_v^2 + (s-p) a_v a_w - r a_w^2}{(a-b)q} + \frac{a_v r_v}{r} + \frac{2 a_v p_w - a_w p_v}{q} \\ a_{vw} &= a_v \frac{a_w + b_w}{a-b} + a_v \left(\frac{q_w}{q} + \frac{r_w}{r} \right) \\ a_{ww} &= \frac{q a_v b_v + (s-p) a_v b_w + r a_w^2}{(a-b)r} + \frac{a_v s_w}{r} + \frac{a_w q_w}{q}. \end{aligned} \tag{13}$$

Equations for b

$$\begin{aligned} b_{vv} &= \frac{r a_w b_w + (p-s) a_v b_w + q b_v^2}{(b-a)q} + \frac{b_w p_v}{q} + \frac{b_v r_v}{r} \\ b_{vw} &= b_w \frac{a_v + b_v}{b-a} + b_w \left(\frac{q_v}{q} + \frac{r_v}{r} \right) \\ b_{ww} &= \frac{r a_w b_w + 2r b_w^2 + (p-s) b_v b_w - q b_v^2}{(b-a)r} + \frac{b_w q_w}{q} + \frac{2 b_w s_v - b_v s_w}{r}. \end{aligned} \tag{14}$$

Equations for p

$$\begin{aligned} p_{vv} &= 2 \frac{r(a_v b_w - a_w b_v) + (s-p) a_v b_v}{(a-b)^2} + \frac{r_v p_v}{r} + \frac{p_v p_w}{q} \\ &+ \frac{\frac{r}{q}(2 q_v a_w - 2 a_v q_w + a_w p_w) - b_v p_v + 2 r_v a_w - 2 a_v (s_v + p_v + r_w) + \frac{p-s}{q}(2 p_v a_w - a_v p_w)}{b-a} \\ p_{vw} &= 2(s-p) \frac{a_v b_w}{(a-b)^2} - \frac{b_w p_v + (2 s_w + p_w) a_v}{b-a} + p_v \left(\frac{q_w}{q} + \frac{r_w}{r} \right) \\ p_{ww} &= 2 \frac{q(a_w b_v - a_v b_w) + (s-p) a_w b_w}{(a-b)^2} + \frac{(p-s) b_w p_v - q b_v p_v - 2 r s_w a_w - r a_w p_w}{(b-a)r} \\ &\quad + \frac{p_v s_w}{r} + \frac{q_w p_w}{q}. \end{aligned} \tag{15}$$

Equations for s

$$
\begin{aligned}
s_{vv} &= 2\frac{r(a_w b_v - a_v b_w) + (p-s)a_v b_v}{(a-b)^2} + \frac{(s-p)a_v s_w - r a_w s_w - 2q p_v b_v - q b_v s_v}{(a-b)q} \\
&\quad + \frac{p_v s_w}{q} + \frac{r_v s_v}{r} \\
s_{vw} &= 2(p-s)\frac{a_v b_w}{(a-b)^2} - \frac{a_v s_w + (2p_v + s_v)b_w}{a-b} + s_w\left(\frac{q_v}{q} + \frac{r_v}{r}\right) \\
s_{ww} &= 2\frac{q(a_v b_w - a_w b_v) + (p-s)a_w b_w}{(a-b)^2} + \frac{q_w s_w}{q} + \frac{s_v s_w}{r} \\
&\quad + \frac{\frac{q}{r}(2r_w b_v - 2b_w r_v + b_v s_v) - a_w s_w + 2q_w b_v - 2b_w(p_w + s_w + q_v) + \frac{s-p}{r}(2s_w b_v - b_w s_v)}{a-b}.
\end{aligned}
\tag{16}
$$

Equations for q and r

$$
\begin{aligned}
q r_{vv} + r q_{vv} &= 2(p-s)\frac{(p-s)a_w b_w + q(a_v b_w - a_w b_v)}{(a-b)^2} + q\frac{r_v}{r}\frac{q b_v + (s-p)b_w}{a-b} \\
&\quad + (s-p)\frac{2a_w s_w + 2b_w p_w + b_w q_v}{a-b} + r\frac{(a_w - 2b_w)q_w}{a-b} \\
&\quad + q\frac{a_w r_w + b_v(2p_w + 2s_w + q_v) - 2b_w(r_w + p_v + s_v)}{a-b} \\
&\quad + \frac{r}{q}q_w^2 + \frac{q}{r}s_w r_v - q_w r_w + s_w(2p_w + q_v) \\
q_{vw} &= (s-p)\frac{q a_v b_v + (s-p)a_v b_w + r a_w b_w}{r(a-b)^2} + \frac{q_v q_w}{q} + \frac{p_v s_w}{r} \\
&\quad + \frac{a_v(r q_w + q r_w) + (s-p)(a_v s_w + b_w p_v) + r a_w s_w + q p_v b_v}{r(a-b)} \\
r_{vw} &= (p-s)\frac{r a_w b_w + (p-s)a_v b_w + q a_v b_v}{q(a-b)^2} + \frac{r_v r_w}{r} + \frac{p_v s_w}{q} \\
&\quad + \frac{b_w(r q_v + q r_v) + (p-s)(a_v s_w + b_w p_v) + r a_w s_w + q p_v b_v}{q(b-a)} \\
q r_{ww} + r q_{ww} &= 2(s-p)\frac{(s-p)a_v b_v + r(a_v b_w - a_w b_v)}{(a-b)^2} + r\frac{q_w}{q}\frac{r a_w + (p-s)a_v}{b-a} \\
&\quad + (p-s)\frac{2b_v p_v + 2a_v s_v + a_v r_w}{b-a} + q\frac{(b_v - 2a_v)r_v}{b-a} \\
&\quad + r\frac{b_v q_v + a_w(2s_v + 2p_v + r_w) - 2a_v(q_v + s_w + p_w)}{b-a} \\
&\quad + \frac{q}{r}r_v^2 + \frac{r}{q}p_v q_w - r_v q_v + p_v(2s_v + r_w).
\end{aligned}
\tag{17}
$$

Note that there are only two relations among the second derivatives $q_{vv}, r_{vv}, q_{ww}, r_{ww}$. These formulae are completely symmetric under the identification $v \leftrightarrow w, a \leftrightarrow b, p \leftrightarrow s, q \leftrightarrow r$. It can be verified that equations (13)–(17) are in involution and their general solution depends, modulo the coordinate transformations $v = \varphi(\tilde{v}), w = \psi(\tilde{w})$, on 15 arbitrary constants. Thus, we have established the existence of a 15-parameter family of integrable systems of the form (8).

Once the integrability conditions (13)–(17) are satisfied, the general solution of the involutive system (11), (12) for φ^i and w will depend on $2n$ arbitrary functions of a single variable (indeed, one can formulate the Goursat problem for this system specifying φ^i along the R^i-coordinate line and specifying the restriction of w to each of the coordinate lines). This has to be considered up to reparametrizations of the form $R^i \to f^i(R^i)$. Thus, the general n-component reduction depends on n essential functions of a single variable. This justifies the definition of integrability given in the introduction. The system (11), (12) governing n-component reductions will be called the generalized Gibbons–Tsarev system (it was derived by Gibbons and Tsarev [17] in the context of the dispersionless KP equation).

Remark. Rewriting equations $(13)_2$ and $(14)_2$ in the form $\mathrm{d}\ln(qr) = \Omega$, where

$$\Omega = \left(\frac{b_{vw}}{b_w} + \frac{a_v + b_v}{a-b}\right)\mathrm{d}v + \left(\frac{a_{vw}}{a_v} + \frac{a_w + b_w}{b-a}\right)\mathrm{d}w$$

(we assume $a_v \neq 0$, $b_w \neq 0$), one obtains the condition $\mathrm{d}\Omega = 0$ which involves the matrix A only. Obviously, the same condition holds for an arbitrary matrix in the linear pencil $\alpha A + \beta B$ (written in the diagonal form). The object $\mathrm{d}\Omega$ first appeared in [10, 11] as one of the basic reciprocal invariants of two-component hydrodynamic-type systems.

3. Conservation laws

In this section we prove the following

Theorem 1. *Any two-component (2+1)-dimensional system of hydrodynamic type which passes the integrability test necessarily possesses three conservation laws of hydrodynamic type and, hence, is symmetrizable in Godunov's sense [19].*

This explains the observation made in our recent publication [14]. To obtain the proof we first transform the system into the form (8). Looking for conservation laws in the form

$$h(v,w)_t + g(v,w)_x + f(v,w)_y = 0$$

one readily obtains

$$g_v = ah_v \qquad g_w = bh_w$$

and

$$f_v = ph_v + rh_w \qquad f_w = qh_v + sh_w.$$

The consistency condition $g_{vw} = g_{wv}$ implies

$$h_{vw} = \frac{a_w}{b-a}h_v + \frac{b_v}{a-b}h_w \tag{18}$$

while the consistency condition $f_{vw} = f_{wv}$ results in

$$p_w h_v + p\left(\frac{a_w}{b-a}h_v + \frac{b_v}{a-b}h_w\right) + r_w h_w + rh_{ww}$$
$$= s_v h_w + s\left(\frac{a_w}{b-a}h_v + \frac{b_v}{a-b}h_w\right) + q_v h_v + qh_{vv}.$$

The last formula can be rewritten in the form

$$\begin{aligned} h_{vv} &= \frac{1}{q}\left(\frac{s-p}{a-b}a_w + p_w - q_v\right)h_v + \frac{l}{q} \\ h_{ww} &= \frac{1}{r}\left(\frac{s-p}{a-b}b_v + s_v - r_w\right)h_w + \frac{l}{r} \end{aligned} \tag{19}$$

where the equations for the auxiliary variable l can be obtained from the compatibility conditions $(h_{vv})_w = (h_{vw})_v$ and $(h_{ww})_v = (h_{vw})_w$:

$$
\begin{aligned}
l_v &= \left(\frac{r_v}{r} + \frac{b_v}{a-b}\right) l - \frac{2b_v p_v(b-a) + 2a_v b_v(p-s) + 4ra_w b_v - ra_v b_w}{(a-b)^2} h_w \\
&\quad - \left\{ \left(a_v b_v q + a_w b_w r + (a-b) r a_w \frac{q_w}{q} + (a_w b_v - a_v b_w)(p-s) \right.\right. \\
&\quad \left.\left. + (b-a) a_w (s_v - r_w) + (a-b) a_v s_w - r a_w^2 \right) \Big/ (a-b)^2 \right\} h_v \\
l_w &= \left(\frac{q_w}{q} + \frac{a_w}{b-a}\right) l - \frac{2a_w s_w(a-b) + 2a_w b_w(s-p) + 4qa_w b_v - qa_v b_w}{(a-b)^2} h_v \\
&\quad - \left\{ \left(a_v b_v q + a_w b_w r + (b-a) q b_v \frac{r_v}{r} + (a_w b_v - a_v b_w)(s-p) \right.\right. \\
&\quad \left.\left. + (a-b) b_v (p_w - q_v) + (b-a) b_w p_v - q b_v^2 \right) \Big/ (a-b)^2 \right\} h_w .
\end{aligned}
\tag{20}
$$

One can verify that the compatibility conditions $l_{vw} = l_{wv}$ are satisfied identically by virtue of (13)–(17). Thus, the system of equations (18)–(20) for conservation laws is in involution and its solution space is three-dimensional.

4. Pseudopotentials

In this section we prove that any integrable system (8) possesses a scalar pseudopotential depending, in some cases, on the auxiliary parameter λ. We begin with some supporting examples.

Example 5. The linearly degenerate system [9, 30, 34]

$$m_t + n_x = 0 \qquad n_t + mn_x - nm_x + m_y = 0$$

from example 3 possesses the pseudopotential

$$\psi_t = -(\lambda + m)\psi_x \qquad \psi_y = (\lambda^2 + \lambda m + n)\psi_x.$$

We emphasize that the parameter λ is essential here, allowing one to recover the full system for m, n from the consistency condition $\psi_{ty} = \psi_{yt}$.

Example 6. The dispersionless KP equation, $(u_t - uu_x)_x = u_{yy}$, rewritten in two-component form

$$u_y = w_x \qquad w_y = u_t - uu_x$$

possesses the pseudopotential

$$\psi_t = \tfrac{1}{3}\psi_x^3 + u\psi_x + w \qquad \psi_y = \tfrac{1}{2}\psi_x^2 + u$$

see [36].

Example 7. The Boyer–Finley equation, $u_{tt} = (\ln u)_{xy}$, rewritten in two-component form

$$u_t = w_y \qquad w_t = u_x/u$$

possesses the pseudopotential

$$\psi_t = \ln u - \ln \psi_y \qquad \psi_x = w - \frac{u}{\psi_y}.$$

Further examples of integrable (2+1)-dimensional equations possessing pseudopotentials of the above type can be found in [21, 31, 36]. It is a remarkable fact that in all examples constructed in [31] the existence of such pseudopotentials manifests the equivalence of the corresponding (2+1)-dimensional system to a pair of commuting (1+1)-dimensional hydrodynamic chains.

In the general case of system (8) we look for a pseudopotential in the form

$$\psi_t = f(\psi_y, v, w) \qquad \psi_x = g(\psi_y, v, w). \tag{21}$$

Writing out the consistency condition $\psi_{tx} = \psi_{xt}$, expressing v_t, w_t by virtue of (8) and equating to zero coefficients at v_x, v_y, w_x, w_y, one arrives at the following expressions for the first derivatives f_v, f_w, f_ξ and g_ξ (we adopt the notation $\xi \equiv \psi_y$):

$$\begin{aligned} &f_v = -a g_v \qquad f_w = -b g_w \\ &f_\xi = \frac{b\left(p + r\frac{g_w}{g_v}\right) - a\left(s + q\frac{g_v}{g_w}\right)}{a-b} \end{aligned} \tag{22}$$

and

$$g_\xi = \frac{s + q\frac{g_v}{g_w} - p - r\frac{g_w}{g_v}}{a-b} \tag{23}$$

The consistency conditions of equations (22) imply the following expressions for the second partial derivatives g_{vw}, g_{vv}, g_{ww}:

$$\begin{aligned} g_{vw} &= \frac{a_w}{b-a} g_v + \frac{b_v}{a-b} g_w \\ g_{vv} &= \frac{g_v\left[g_w^2(r(b_v - a_v) + (a-b)r_v) + g_v g_w((a-b)p_v + (s-p)a_v - r a_w) + q a_v g_v^2\right]}{(a-b) r g_w^2} \\ g_{ww} &= \frac{g_w\left[g_v^2(q(a_w - b_w) + (b-a)q_w) + g_v g_w((b-a)s_w + (p-s)b_w - q b_v) + r b_w g_w^2\right]}{(b-a) q g_v^2}. \end{aligned} \tag{24}$$

The compatibility conditions of equations (23), (24) for g, namely, the conditions $g_{\xi vv} = g_{vv\xi}$, $g_{\xi vw} = g_{vw\xi}$, etc, are of the form $P(g_v, g_w) = 0$, where P denotes a rational expression in g_v, g_w whose coefficients are functions of a, b, p, q, r, s and their partial derivatives up to the second order. Equating all these expressions to zero (they are required to be zero identically in g_v, g_w), one obtains the set of conditions which are necessary and sufficient for the existence of pseudopotentials of the form (21). It is a truly remarkable fact that these conditions *identically* coincide with the integrability conditions (13)–(17). Thus, any system satisfying the integrability conditions (13)–(17) possesses pseudopotentials of the form (21) parametrized by four arbitrary integration constants, indeed, one can arbitrarily prescribe the values of g, g_v, g_w and f at any initial point; the rest is completely determined by the involutive system (23), (24) and (22). Note, however, that the transformation $\psi \to \lambda\psi + \mu x + \nu y + \eta t$ allows one to eliminate all these constants in the general situation (see example 5 where one of these constants survives and is essential).

We have established the following

Theorem 2. *The class of two-component (2+1)-dimensional systems of hydrodynamic type possessing infinitely many hydrodynamic reductions coincides with the class of systems possessing a scalar pseudopotential of the form (21).*

Remark. The pseudopotential (21) readily implies a pseudopotential for the corresponding generalized Gibbons–Tsarev system (11), (12). Indeed, differentiating equations (21) by y and introducing $\xi = \psi_y$, one obtains

$$\xi_t = \partial_y f(\xi, v, w) = f_\xi \xi_y + f_v v_y + f_w w_y \qquad \xi_x = \partial_y g(\xi, v, w) = g_\xi \xi_y + g_v v_y + g_w w_y.$$

Assuming now that ξ, v, w are functions of n Riemann invariants $R^1, \dots, R^n$ which satisfy equations (2), one arrives at

$$\xi_i \lambda^i = f_\xi \xi_i + f_v v_i + f_w w_i \qquad \xi_i \mu^i = g_\xi \xi_i + g_v v_i + g_w w_i.$$

Substituting here $v_i = \varphi^i w_i$, expressions (9) for λ^i and μ^i in terms of φ^i (see section 2), and taking into account formulae (22), (23), one ends up with

$$\xi_i = \frac{(a-b)\varphi^i}{r\varphi^i/g_v - q/g_w} w_i. \tag{25}$$

Equations (25) define a scalar pseudopotential for the generalized Gibbons–Tsarev system (11), (12), i.e., the consistency conditions of (25) imply equations (11), (12).

5. Examples

Equations (13)–(17) are particularly convenient to analyse when the matrix A is *given* (we emphasize that a and b cannot be arbitrary). The corresponding matrix B is then defined up to a natural equivalence

$$B \to \mu B + \nu A + \eta I_2$$

generated by a linear change of the independent variables in equations (8): $\tilde{t} = t, \tilde{x} = x, \tilde{y} = \mu y + \nu x + \eta t$; here μ, ν, η are arbitrary constants. Moreover, one has a freedom of the coordinate transformations $v = \varphi(\tilde{v}), w = \psi(\tilde{w})$ preserving the diagonal form of A. These transformations do not change a, b, p, s and transform q and r according to the formulae

$$\tilde{q} = q\frac{\psi'(\tilde{w})}{\varphi'(\tilde{v})} \qquad \tilde{r} = r\frac{\varphi'(\tilde{v})}{\psi'(\tilde{w})}.$$

The classification results presented below are carried out up to this natural equivalence.

In this section we concentrate on the case when the matrix A is linearly degenerate, i.e., $a_v = b_w = 0$. There are three essentially different cases to consider:

$$\text{Case 1}: A = \begin{pmatrix} w & 0 \\ 0 & v \end{pmatrix} \qquad \text{Case 2}: A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \qquad \text{Case 3}: A = \begin{pmatrix} w & 0 \\ 0 & \beta \end{pmatrix}$$

here α and β are arbitrary constants. Note that without any loss of generality one can set $\alpha = 1, \beta = 0$. Below we restrict ourselves to the symmetric cases 1 and 2, and show that there is a multi-parameter freedom in the formulae for B.

Case 1. Substituting $a = w, b = v$ into the integrability conditions (13)–(17), one obtains the overdetermined system for p, q, r, s which can be explicitly integrated (the integration is fairly straightforward so that we skip the details). Up to the equivalence mentioned above we have

$$p = \frac{f(w)}{w-v} - \alpha w^2 \qquad q = \frac{f(v)}{w-v} \qquad r = \frac{f(w)}{v-w} \qquad s = \frac{f(v)}{v-w} - \alpha v^2$$

where f is a cubic polynomial, $f(z) = \alpha z^3 + \beta z^2 + \gamma z + \delta$, and $\alpha, \beta, \gamma, \delta$ are arbitrary constants. A remarkable property of this example is that *any* matrix in the linear pencil $B + \mu A$ is also linearly degenerate. In the particular case $\alpha = \beta = \gamma = 0, \delta = 1$ one has

$$v_t + wv_x + \frac{1}{w-v}(v_y + w_y) = 0 \qquad w_t + vw_x + \frac{1}{v-w}(v_y + w_y) = 0.$$

This system possesses three conservation laws

$$(v+w)_t + (vw)_x = 0$$

$$(v^2 + vw + w^2)_t + (vw(v+w))_x - (v+w)_y = 0$$

and

$$(v^3 + v^2w + vw^2 + w^3)_t + (vw(v^2 + vw + w^2))_x - (v+w)^2_y = 0.$$

Introducing the variables $m = v + w$, $n = vw$, one can rewrite this system as

$$m_t + n_x = 0 \qquad n_t + mn_x - nm_x + m_y = 0.$$

In this form it was thoroughly investigated in [30] (see also [29]).

Case 2. Here a and b are constants, $a \neq b$. The corresponding equations for p, q, r, s take the form

$$p_{vv} = p_v\left(\frac{p_w}{q} + \frac{r_v}{r}\right) \qquad p_{vw} = p_v\left(\frac{q_w}{q} + \frac{r_w}{r}\right) \qquad p_{ww} = \frac{p_w q_w}{q} + \frac{s_w p_v}{r}$$

$$s_{vv} = \frac{s_v r_v}{r} + \frac{s_w p_v}{q} \qquad s_{vw} = s_w\left(\frac{q_v}{q} + \frac{r_v}{r}\right) \qquad s_{ww} = s_w\left(\frac{s_v}{r} + \frac{q_w}{q}\right)$$

$$qr_{vv} + rq_{vv} = \frac{qr_v^2}{r} - q_v r_v + p_v(2s_v + r_w) + \frac{rq_w p_v}{q}$$

$$qr_{ww} + rq_{ww} = \frac{rq_w^2}{q} - q_w r_w + s_w(2p_w + q_v) + \frac{qr_v s_w}{r}$$

$$q_{vw} = \frac{q_v q_w}{q} + \frac{p_v s_w}{r} \qquad r_{vw} = \frac{r_v r_w}{r} + \frac{p_v s_w}{q}.$$

These equations imply, in particular, that $(p_v/qr)_w = 0, (s_w/qr)_v = 0$, so that, after the appropriate reparametrization $v \to f(v), w \to g(w)$, one can set $p_v = s_w = qr$ (provided $p_v \neq 0, s_w \neq 0$). With this simplification, the above equations reduce to

$$p_v = qr \qquad p_w = q_v \qquad s_v = r_w \qquad s_w = qr$$

along with the following overdetermined system for q and r:

$$\begin{aligned} q_{vv} &= (qr)_w \qquad & q_{vw} &= \frac{q_v q_w}{q} + q^2 r \qquad & q_{ww} &= \frac{q_w^2}{q} + 2qq_v - \frac{q_w r_w}{r} \\ r_{ww} &= (qr)_v \qquad & r_{vw} &= \frac{r_v r_w}{r} + qr^2 \qquad & r_{vv} &= \frac{r_v^2}{r} + 2rr_w - \frac{q_v r_v}{q}. \end{aligned} \tag{26}$$

This system is in involution with the general solution depending on six arbitrary constants. Equations for q_{vw} and r_{vw} yield the Liouville equation for $\ln(qr)$ and the linear wave equation for $\ln(q/r)$, implying the following functional ansatz for these variables:

$$q = \frac{f'(v)^{1/2} g'(w)^{1/2}}{f(v) + g(w)} \frac{m(f(v))}{n(g(w))} \qquad r = \frac{f'(v)^{1/2} g'(w)^{1/2}}{f(v) + g(w)} \frac{n(g(w))}{m(f(v))}. \tag{27}$$

Setting

$$(f')^{3/2} = P(f) \qquad (g')^{3/2} = Q(g) \tag{28}$$

and substituting (27) into the remaining equations (26), we obtain the following four functional-differential equations for $P(f)$, $Q(g)$, $m(f)$, $n(g)$:

$$[P''(f+g)^2 - 4P'(f+g) + 6P]m$$
$$+[4P'(f+g)^2 - 6P(f+g)]m' + 3P(f+g)^2m'' = [2Q'(f+g) - 6Q]n$$
$$[Q''(f+g)^2 - 4Q'(f+g) + 6Q]n$$
$$+[4Q'(f+g)^2 - 6Q(f+g)]n' + 3Q(f+g)^2n'' = [2P'(f+g) - 6P]m$$
$$Pm' + Q'(f+g)n' + Q\left[(f+g)n'' - n'\right] = 0$$
$$Qn' + P'(f+g)m' + P[(f+g)m'' - m'] = 0.$$

These equations yield

$$P(f) = \frac{\alpha f^3 + \beta f^2 + \gamma f + \delta}{m(f)} \qquad Q(g) = \frac{\alpha g^3 - \beta g^2 + \gamma g - \delta}{n(g)}$$

where

$$(\ln m)' = \frac{Af}{\alpha f^3 + \beta f^2 + \gamma f + \delta} \qquad (\ln n)' = -\frac{Ag}{\alpha g^3 - \beta g^2 + \gamma g - \delta}.$$

Here $\alpha, \beta, \gamma, \delta$ and A are arbitrary constants. If $A = 0$ and $m = n = 1$, then both P and Q are cubic polynomials in f and g, implying that equations for f and g can be solved in terms of elliptic functions (this case was considered in [14]).

6. Conclusion

In this paper we gave the characterization of two-component (2+1)-dimensional integrable systems of hydrodynamic type, showing that

- there exists a 15-parameter family of such systems;
- all integrable systems are symmetrizable in Godunov's sense;
- the system is integrable iff it possesses a scalar pseudopotential.

We have also constructed nontrivial explicit examples of integrable two-component (2+1)-dimensional systems of hydrodynamic type for which one of the matrices of the system is linearly degenerate.

The important problem remaining is to clarify the differential geometry of the full set of integrability conditions (13)–(17) expressing them in invariant form in terms of the corresponding matrices A and B.

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